

Polarisation in the Temperature Equilibration of Anisotropic Plasmas

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The polarisation effect on the isotropisation of a magnetized bi-Maxwellian plasma is investigated in the weak coupling approximation. Based on the Rostoker collision operator, which reduces in the magnetic field free limit to the Balescu-Lenard operator, the equipartition frequency ν is evaluated numerically for $B \rightarrow 0$ and analytically for $B \rightarrow \infty$, respectively. This frequency, depending non-linearly on ε and B , describes the approach to thermodynamic equilibrium of the anisotropy parameter $\varepsilon \equiv T_{\perp}/T_{\parallel}$ via $d\varepsilon/dt = (1-\varepsilon)\nu(\varepsilon, B)$. It is found that in both limits dynamical screening makes a negligible contribution to the relaxation process. For finite B a simplified, yet unexploited expression for $\nu(\varepsilon, B)$ is presented, which invokes three new functions.

Key words: Coulomb collisions, Weak coupling theory, Polarisation, Collision frequency.

1. Introduction

As a specific example of a dynamical system far away from thermodynamic equilibrium we consider an anisotropic plasma which is initially distributed according to a bi-Maxwellian and in which temperature equalization takes place due to Coulomb interactions. This system appears to be of interest not only because of its experimental evidence [1], but also because of its potential impact on general plasma transport in situation of weak collisionality. In earlier works several aspects of this relaxation process have been investigated. In [2] a nonlinear analysis of the equipartition frequency ν in the magnetic field free region has been made under the assumption that the relaxation takes place within the class of bi-Maxwellian distributions. This latter assumption has been checked by computer simulations and was found to be well satisfied, when $T_{\perp} > T_{\parallel}$, where T_{\perp} (T_{\parallel}) is the temperature perpendicular (parallel) to a preference direction \hat{e} . In cases of $T_{\parallel} > T_{\perp}$, however, the true relaxation turned out to be slower than predicted by the bi-

Maxwellian model so that the equipartition frequency ν obtained within the bi-Maxwellian assumption can be considered as an upper bound to the true relaxation frequency. In [3–5] the magnetic field effect on ν was investigated and a logarithmic decrease of ν with increasing B was found in the region $b < r_e < \lambda_D$, where b is the classical distance of closest approach (Landau length), r_e is the cyclotron radius and λ_D is the Debye length of the considered species. In addition, the weak coupling approximation, in which only small angle scattering events are treated correctly, was found to be an adequate description as long as $b < r_e$. For ultra strong magnetic field strengths, when $r_e < b$, O’Neil [6] and O’Neil and Hjorth [7] argued that large angle binary collisions are dominant and hence developed a strong coupling theory which could be confirmed experimentally [8]. For more moderate magnetic fields, however, when weak coupling is applicable, the relaxation speed may also be influenced by polarisation. Investigations, so far, have ignored screening to keep the already complex analysis simpler. Since, however, there is no obvious argument why polarisation effects should be negligible, we decided to analyse this aspect in more detail. The present paper gives an account of this investigation.

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2. Anisotropy Relaxation without Magnetic Field

In this section we first neglect magnetic field influences. We start with the electron distribution of a plasma which is initially distributed according to a bi-Maxwellian

$$f(\mathbf{v}) = \left(\frac{m}{2\pi}\right)^{3/2} \frac{1}{\sqrt{k_B T_{\parallel}}} \frac{1}{k_B T_{\perp}} \exp \left\{ -\frac{m}{2k_B T_{\perp}} \left[(\mathbf{v} \cdot \hat{e})^2 \left(\frac{T_{\perp}}{T_{\parallel}} - 1 \right) + v^2 \right] \right\}, \quad (1)$$

where m is the electron mass and k_B the Boltzmann constant. The relaxation process caused by Coulomb collisions among the electrons is then described by the Lenard-Balescu collision operator [9]

$$\left(\frac{\partial f}{\partial t} \right)_c = -\frac{\partial}{\partial \mathbf{v}} \cdot \int \tilde{Q}(\mathbf{v}, \mathbf{v}') \cdot (\partial \mathbf{v} - \partial \mathbf{v}') f(\mathbf{v}) f(\mathbf{v}') d^3 v', \quad (2)$$

where

$$\tilde{Q}(\mathbf{v}, \mathbf{v}') = -\frac{\omega_e^4}{2n(2\pi)^4} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \left\{ \frac{\exp[i(\omega - \mathbf{k} \cdot \mathbf{v})s] \exp[i(\omega - \mathbf{k} \cdot \mathbf{v}')s']}{|\varepsilon_D(\mathbf{k}, \omega)|^2} \frac{\mathbf{k} \mathbf{k}}{k^4} \right\} ds' ds d\omega d^3 k \quad (3)$$

and

$$\varepsilon_D(\mathbf{k}, \omega) = 1 - \frac{\omega_e^2 m}{2k_B T_{\parallel}} \frac{1}{\frac{T_{\perp}}{T_{\parallel}}(k_x^2 + k_y^2) + k_z^2} Z' \left[\sqrt{\frac{m}{2k_B T_{\parallel} \left[\frac{T_{\perp}}{T_{\parallel}}(k_x^2 + k_y^2) + k_z^2 \right]}} \omega \right]. \quad (4)$$

ε_D , the dielectric function, represents dynamical screening. Z in (4) is the plasma dispersion function and ω_e the plasma frequency. To evaluate the time behaviour of the anisotropy parameter $\varepsilon = T_{\perp}/T_{\parallel}$ we separate the total kinetic energy of the plasma $W = \int n \frac{m}{2} v^2 f(\mathbf{v}) d^3 v$ into the part $W_{\parallel} = n \frac{m}{2} \int (\mathbf{v} \cdot \hat{e})^2 f(\mathbf{v}) d^3 v$ parallel to \hat{e} and a part $W_{\perp} = n \frac{m}{2} \int (\hat{e} \times \mathbf{v})^2 f(\mathbf{v}) d^3 v$ perpendicular to it. Using energy conservation we get

$$\frac{d\varepsilon}{dt} = -\frac{1}{n} \frac{(1+2\varepsilon)^3}{3k_B T_0} \frac{dW_{\parallel}}{dt}. \quad (5)$$

T_0 means the equilibrium temperature at the end of the relaxation process. The time derivative of the kinetic energy along \hat{e} is computed by means of a partial integration and is given by

$$\frac{dW_{\parallel}}{dt} = n \iint m(\mathbf{v} \cdot \hat{e}) \hat{e} \cdot \tilde{Q}(\mathbf{v}, \mathbf{v}') \cdot (\partial \mathbf{v} - \partial \mathbf{v}') f(\mathbf{v}) f(\mathbf{v}') d^3 v' d^3 v. \quad (6)$$

To do the velocity integrations we make the substitutions

$$\mathbf{u} = \mathbf{v} - \mathbf{v}', \quad (7)$$

$$\mathbf{t} = \frac{1}{2}(\mathbf{v} + \mathbf{v}'), \quad (8)$$

assume bi-Maxwellians throughout the evolution and arrive at

$$\begin{aligned} \frac{dW_{\parallel}}{dt} = & \frac{\omega_e^4 m}{4(2\pi)^4} \left(1 - \frac{T_{\parallel}}{T_{\perp}}\right) \int_{-\infty}^{\infty} \int \left\langle \frac{(\mathbf{k} \cdot \hat{e})^2}{k^4} \frac{1}{|\varepsilon_D(\mathbf{k}, \omega)|^2} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \left\{ \left\{ 2 - \frac{k_B T_{\parallel}}{m} (\mathbf{k} \cdot \hat{e})^2 (s-s') [(s-s') + (s+s')] \right\} \right. \right. \\ & \cdot \exp \left\{ i(s+s')\omega - \frac{k_B T_{\parallel}}{4m} \left[\frac{T_{\perp}}{T_{\parallel}} k^2 + \left(1 - \frac{T_{\perp}}{T_{\parallel}}\right) (\mathbf{k} \cdot \hat{e})^2 \right] [(s+s')^2 + (s-s')^2] \right\} ds' ds \right\rangle d^3 k d\omega. \quad (9) \end{aligned}$$

Using an equivalent substitution we can evaluate the s - and s' -integrations. Finally the result is set into (5) for the anisotropy parameter, which becomes

$$\frac{d\varepsilon}{d\tau} = \frac{1}{2\pi\sqrt{\pi\ln A}} (1+2\varepsilon)^{3/2} (1-\varepsilon) \quad (10)$$

$$\cdot \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \left\langle \frac{(\mathbf{k} \cdot \hat{\mathbf{e}})^2}{k^4} \frac{\mathbf{k}^2 - (\mathbf{k} \cdot \hat{\mathbf{e}})^2}{[\varepsilon \mathbf{k}^2 + (1-\varepsilon)(\mathbf{k} \cdot \hat{\mathbf{e}})^2]^{3/2}} \frac{\exp(-x^2)}{\left| \varepsilon_D \left(k_D \mathbf{k}, \omega_e \sqrt{\frac{3}{1+2\varepsilon}} \sqrt{\varepsilon \mathbf{k}^2 + (1-\varepsilon)(\mathbf{k} \cdot \hat{\mathbf{e}})^2} \right) \right|^2} \right\rangle dx d^3k,$$

where $t = \frac{2\sqrt{3}}{v_{\text{Spitzer}}}$ is a time normalized by Spitzer's momentum exchange frequency [10], $A = \frac{4\pi}{3} n \lambda_D^3$ and k_D is the inverse Debye-length. By neglecting polarisation effects, i.e. by setting the dielectric function equal to one, we arrive at the result of Schamel et al. [2]. To compute the polarisation effect we introduce the relaxation frequency by means of the definition

$$\frac{d\varepsilon}{dt} = v(\varepsilon) (1-\varepsilon) \quad (11)$$

and evaluate $v(\varepsilon)$ numerically. In Fig. 1 v is plotted as a function of the anisotropy parameter. A comparison with the corresponding expression (Fig. 3 of [2]) obtained without polarisation effects ($|\varepsilon_D|^2 \equiv 1$) shows that dynamical screening affects the relaxation process by less than 0.3 percent for $20^{-1} \leq \varepsilon \leq 5$, only.

3 Anisotropy Relaxation Influenced by a Magnetic Field

Now we take magnetic field effects into account and describe Coulomb collisions by means of the Rostoker collision operator [11]:

$$\begin{aligned} \left(\frac{\partial f}{\partial t} \right)_c &= \frac{1}{2\pi} \left(\frac{\omega_e}{2\pi} \right)^4 \frac{\partial}{\partial v} \\ &\cdot \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \exp \left\langle i \left\{ [\omega - k_z v_z] s - \frac{1}{\Omega} [(v_y k_x - k_y v_x) (\cos(\Omega s) - 1) + (v_x k_x + v_y k_y) \sin(\Omega s)] \right. \right. \\ &+ [\omega - k_z v'_z] s' - \frac{1}{\Omega} [(v'_y k_x - k_y v'_x) (\cos(\Omega s') - 1) + (v'_x k_x + v'_y k_y) \sin(\Omega s')] \left. \left. \right\} \right\rangle \\ &\cdot \frac{\mathbf{k} \mathbf{k}}{k^4} (\partial v - \partial v') \frac{f(v_{\perp}, v_{\parallel}) f(v'_{\perp}, v'_{\parallel})}{|\varepsilon_D(B, \mathbf{k}, \omega)|^2} d^3v' ds' ds d\omega d^3k. \end{aligned} \quad (12)$$

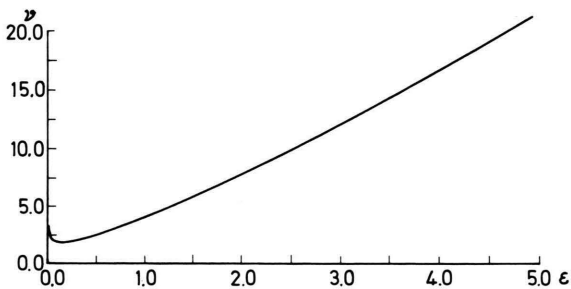


Fig. 1. The equipartition frequency $v(\varepsilon)$ as a function of the anisotropy parameter $\varepsilon = T_{\perp}/T_{\parallel}$.

Ω is the gyrofrequency of an electron in an externally given magnetic field B and ε_D , which is evaluated for bi-Maxwellians [12], is given by

$$\varepsilon(B, \mathbf{k}, \omega) = 1 + \frac{\omega_e^2}{k^2} \left\{ \frac{m}{k_B T_{\parallel}} + \frac{m}{|k_{\parallel}|} \sqrt{m/2 k_B T_{\parallel}} \exp \left[-\frac{k_B T_{\perp}}{m} \left(\frac{k_{\perp}}{\Omega} \right)^2 \right] \right. \\ \left. \cdot \sum_{n=-\infty}^{\infty} \left[I_n \left(\frac{k_B T_{\perp}}{m} \left(\frac{k_{\perp}}{\Omega} \right)^2 \right) \left(\frac{n\Omega}{k_B T_{\perp}} + \frac{\omega - n\Omega}{k_B T_{\parallel}} \right) Z \left(\sqrt{\frac{m}{2 k_B T_{\parallel}}} \frac{\omega - n\Omega}{|k_{\parallel}|} \right) \right] \right\}. \quad (13)$$

The time behaviour of the anisotropy parameter is investigated in the same manner as in the magnetic free field case. We get the expression [13]

$$\frac{d\varepsilon}{dt} = \frac{1}{12 n v_{Te}^2} (1 + 2\varepsilon)^2 \left(\frac{\omega_e}{2\pi} \right)^4 \int_{-\infty}^{\infty} \int \frac{k_z^2}{k^4 |\varepsilon_D(B, \mathbf{k}, \omega)|^2} f_{\Omega} \left(\frac{3\varepsilon}{1+2\varepsilon} T_0, \frac{3T_0}{1+2\varepsilon}, \mathbf{k}, \omega \right) d\omega d^3k. \quad (14)$$

Here we have introduced the function

$$f_{\Omega}(T_{\perp}, T_{\parallel}, \omega) = \frac{32}{\Omega^2} \left(1 - \frac{T_{\parallel}}{T_{\perp}} \right) \left\{ \left[\text{ec} \left(2 \frac{k_B T_{\parallel}}{m} \frac{k_z^2}{\Omega^2}, 2 \frac{k_B T_{\perp}}{m} \frac{k_{\perp}^2}{\Omega^2}, 2 \frac{\omega}{\Omega} \right) \right]^2 \right. \\ \left. - 4 \frac{k_B T_{\parallel}}{m} \frac{k_z^2}{\Omega^2} \left[\text{exxc} \left(2 \frac{k_B T_{\parallel}}{m} \frac{k_z^2}{\Omega^2}, 2 \frac{k_B T_{\perp}}{m} \frac{k_{\perp}^2}{\Omega^2}, 2 \frac{\omega}{\Omega} \right) \text{ec} \left(2 \frac{k_B T_{\parallel}}{m} \frac{k_z^2}{\Omega^2}, 2 \frac{k_B T_{\perp}}{m} \frac{k_{\perp}^2}{\Omega^2}, 2 \frac{\omega}{\Omega} \right) \right. \right. \\ \left. \left. + \left(\text{exs} \left(2 \frac{k_B T_{\parallel}}{m} \frac{k_z^2}{\Omega^2}, 2 \frac{k_B T_{\perp}}{m} \frac{k_{\perp}^2}{\Omega^2}, 2 \frac{\omega}{\Omega} \right) \right)^2 \right] \right\} \quad (15)$$

with the abbreviations

$$\text{ec}(a, b, c) = \int_0^{\infty} \exp(-at^2 - b \sin^2 t) \cos(ct) dt, \quad (16)$$

$$\text{exs}(a, b, c) = \int_0^{\infty} t \exp(-at^2 - b \sin^2 t) \sin(ct) dt, \quad (17)$$

$$\text{exxc}(a, b, c) = \int_0^{\infty} t^2 \exp(-at^2 - b \sin^2 t) \cos(ct) dt. \quad (18)$$

Note that these functions are not independent of each other because

$$\text{exs}(a, b, c) = -\frac{\partial}{\partial c} \text{ec}(a, b, c), \quad (19)$$

$$\text{exxc}(a, b, c) = -\frac{\partial}{\partial a} \text{ec}(a, b, c). \quad (20)$$

For an arbitrary magnetic field these equations are still unexploited. Only two special cases are discussed here: In the magnetic field free limit $B \rightarrow 0$ the equations reduce to the result of Section 2. In the opposite limit of an ultra strong magnetic field ($\omega_0 \ll |\Omega|$, where ω_0 is the Landau frequency) we obtain, by using a perpendicular cut-off in k -Space [5], $v(\Omega, \varepsilon) \rightarrow \sqrt{\frac{3(1+2\varepsilon)}{\pi^3}} \frac{A^4}{2\Omega^4}$. This result coincides with the corresponding one of vanishing polarisation [4]. We, hence, conclude that at least in these two limits polarisation makes a negligible contribution to the anisotropy relaxation process.

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